M.Sc.- I (Mathematics) (New CBCS Pattern) Semester-I

PSCMTH04 - Linear Algebra

P. Pages : 2 **GUG/S/25/13740**

Time : Three Hours Max. Marks : 100

Notes : 1. Solve all the **five** questions.
2. Each question carries equal marks.

UNIT-I

1. a) Prove that: In any vector space V, the following statements are true. **10**

i) $0x = 0$, for each $x \in V$

ii) $(-a)x = -(ax) = a(-x)$, for each $a \in F$ and each $x \in V$.

iii) $a0 = 0$, for each $a \in F$.

b) Let S be a linearly independent subset of a vector space V, & let v be a vector in V that is not in S. Then prove that $S \cup \{v\}$ is linearly dependent if only of $v \in \text{span}(S)$. **10**

OR

c) Prove that : If a vector space V is generated by a finite subset S then some subset of S is a basis for V. Hence, V has a finite basis. **10**

d) Prove that : Let W be a subspace of a finite dimensional vector space V. Then W is finite dimensional & $\dim(W) \le \dim(V)$. Moreover, if $\dim(W) \le \dim(V)$ then $V = W$. **10**

UNIT-II

2. a) State and prove dimension theorem. **10**

b) Let V and W be vector spaces over F and suppose that $\{v_1, v_2, ..., v_n\}$ is a basis of V. Prove that for $w_1, w_2, ..., w_n$ in W, there exists exactly one linear transformation $T: V \to W$ such that $T(v_i) = w_i$, for $i = 1, 2, ..., n$. **10**

OR

c) Let T be an invertible linear transformation from V to W. Then prove that V is finite dimensional if and only if W is finite dimensional. In this case, show that $\dim V = \dim W$. **10**

d) Let V and W be finite-dimensional vector spaces (over the same field). Then prove that V is isomorphic to W if and only if $\dim V = \dim W$. **10**

UNIT-III

3. a) Find the eigenvectors of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and prove that A is diagonalizable. **10**

b) Prove that, the characteristic polynomial of any diagonalizable linear operator splits. **10**

OR

c) Let $A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$ then show that A is diagonalizable and find a matrix Q such that $Q^{-1}AQ$ is a diagonal matrix. Also, find A^n. **10**

d) State and prove the Cayley-Hamilton theorem. **10**

UNIT-IV

4. a) Let V be an inner product space over F. Then prove that for all $x, y \in V$ and $c \in F$, the following statements are true: **10**

i) $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$

ii) $\|x + y\| \leq \|x\| + \|y\|$

b) Let V be a finite-dimensional inner product space over F, and let $g : V \to F$ be a linear transformation. Prove that there exists a unique vector $y \in V$ such that $g(x) = \langle x, y\rangle$, for all $x \in V$. **10**

OR

c) Let T be a linear operator on a finite dimensional inner product space of V. suppose that the characteristic polynomial of T splits. Then prove that there exists an orthonormal basis β of V such that the matrix $[T]_\beta$ is upper triangular. **10**

d) Let T be a linear operator on a finite dimensional complex inner product space V. Then prove that T is normal if and only if there exists an orthonormal basis for V consisting of eigen vectors of T. **10**

5. a) Define vector space. **5**

b) Let V and W be vector spaces and let $T : V \to W$ be linear. Then prove that T is one-one if and only if $N(T) = \{0\}$. **5**

c) Find the eigenvalues and of the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. **5**

d) Let $A \in M_{m\times n}(F)$ then prove that rank $(A^* A) = \text{rank}(A)$. **5**

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